MA 3046

Matrix Analysis

Exam I - Quarter II - AY 2003-2004

Instructions: Work all problems. Show appropriate intermediate computations for full credit. Calculators and one page of notes ($8\frac{1}{2}$ by 11 inches, both sides) permitted. *Read the questions carefully.*

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1. (30 points) Using the *modified* Gram-Schmidt method, find a reduced $\mathbf{Q}\,\mathbf{R}$ factoriztion of the matrix

$$\begin{bmatrix} -1 & 6 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

solution:

Note that for **hand** calculation, it's probably better to wait until you've found a full set of orthogonal vectors before you normalize. (Avoids lots of nasty square roots in the calculations.) But we'll follow the modified algorthim:

The modifiedGram-Schmidt algorithm is:

(1) Let $\mathbf{v}^{(j)} = \mathbf{a}^{(j)}$, $j = 1, 2, \ldots, m$

(2) For $j = 2, \ldots, n$,

Form: $\mathbf{q}^{(j)} = \mathbf{v}^{(j)} / \| \mathbf{v}^{(j)} \| \implies r_{jj} = \| \mathbf{v}^{(j)} \|$

For $k = j+1, \ldots, m$

$$\mathbf{v}^{(k)} = \mathbf{v}^{(k)} - r_{jk}\mathbf{q}^{(j)}$$

end

where $r_{jk} = {\mathbf{q}^{(j)}}^H \, \mathbf{v}^{(k)}$

In this problem,

$$\mathbf{v}^{(i)} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v}^{(2)} = \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \text{and} \quad \mathbf{v}^{(3)} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

and so, for $j = 1$, $r_{11} = \| \mathbf{v}^{(1)} \| = \sqrt{(-1)^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2.$

solution:

Therefore

$$\mathbf{q}^{(1)} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Then, for $k = 2$

$$r_{12} = \mathbf{q}^{(1)^H}\mathbf{v}^{(2)} = [\ -\tfrac{1}{2} \quad \tfrac{1}{2} \ \tfrac{1}{2} \quad \tfrac{1}{2}\ ] \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix} = -2$$

and so, removing any component in the direction of $\mathbf{q}^{(1)}$ from $\mathbf{v}^{(2)}$, we have

$$\mathbf{v}^{(2)} = \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix} - (-2) \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

Similarly, for $k = 3$,

$$r_{13} = \mathbf{q}^{(1)^H}\mathbf{v}^{(3)} = [\ -\tfrac{1}{2} \quad \tfrac{1}{2} \ \tfrac{1}{2} \quad \tfrac{1}{2}\ ] \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = 1$$

and so, removing any component in the direction of $\mathbf{q}^{(1)}$ from $\mathbf{v}^{(3)}$, we have

$$\mathbf{v}^{(3)} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} - (1) \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

solution:

Next, for $j = 2$, $r_{22} = \| \mathbf{v}^{(2)} \| = \sqrt{5^2 + 3^2 + 1^2 + 1^2} = \sqrt{36} = 6$, and so

$$\mathbf{q}^{(2)} = \begin{bmatrix} \frac{5}{6} \\ \frac{3}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$$

Now, for $k = 3$,

$$r_{23} = \mathbf{q}^{(2)^H} \mathbf{v}^{(3)} = [\begin{array}{cccc} \frac{5}{6} & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} \end{array}] \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} = -1$$

and so, removing any component in the direction of $\mathbf{q}^{(2)}$ from $\mathbf{v}^{(3)}$, we have

$$\mathbf{v}^{(3)} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} - (-1) \begin{bmatrix} \frac{5}{6} \\ \frac{3}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{3}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Finally, for $j = 3$,

$$\begin{aligned} r_{33} = \| \mathbf{v}^{(3)} \| &= \sqrt{\left(-\frac{1}{3}\right)^2 + \left(-\frac{3}{3}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{1}{3}\right)^2}, \\ &= \sqrt{\frac{36}{9}} = \sqrt{4} = 2 \end{aligned}$$

and so

$$\mathbf{q}^{(3)} = \begin{bmatrix} \frac{1}{6} \\ -\frac{3}{6} \\ \frac{5}{6} \\ -\frac{1}{6} \end{bmatrix}$$

solution:

So, in summary

$$\mathbf{A} = \mathbf{Q}\,\mathbf{R} = \begin{bmatrix} -\frac{1}{2} & \frac{5}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{3}{6} & -\frac{3}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

This result can be verified by direct multiplication.

2. (35 points) Consider the Singular Value Decomposition:

$$\mathbf{A} = \begin{bmatrix} 3.8 & 3.4 \\ 1.0 & 3.0 \\ -0.2 & 1.4 \\ 0.2 & -1.4 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{2} \\ \frac{3}{6} & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}^H$$

a. Using this information, solve the system

$$\mathbf{A}\,\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$$

solution:

First, since $\mathbf{A}$ is 4×2, we know this should be solved as a least squares problem. Furthermore, we know that, since all the singular values of $\mathbf{A}$ are nonzero, then $Col(\mathbf{U}) = Col(\mathbf{A})$. Therefore $\mathbf{x}$ will provide the least squares solution if and only if

$$\mathbf{U}^H(\mathbf{Ax} - \mathbf{b}) = 0 \quad \Longrightarrow \quad \mathbf{U}^H\left(\mathbf{U}\boldsymbol{\Sigma}\,\mathbf{V}^H\mathbf{x} - \mathbf{b}\right) = 0$$
$$\Longrightarrow \quad \boldsymbol{\Sigma}\,\mathbf{V}^H\mathbf{x} = \mathbf{U}^H\mathbf{b}$$

For this problem, this means

$$\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}^H \mathbf{x} = \begin{bmatrix} \frac{5}{6} & \frac{3}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

But since the first matrix on the left is diagonal, we immediately have

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}^H \mathbf{x} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -1 \end{bmatrix}$$

Finally, since $\mathbf{V}$ is unitary

$$\mathbf{x} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} \\ \frac{13}{15} \end{bmatrix} = \begin{bmatrix} -\frac{9}{15} \\ \frac{13}{15} \end{bmatrix}$$

b. Does the solution $\mathbf{x}$ computed in part a. above exactly satisfy the given system of equations? If not, *briefly* explain whether this is a contradiction or not.

solution:

Direct computation shows that

$$\begin{bmatrix} 3.8 & 3.4 \\ 1.0 & 3.0 \\ -0.2 & 1.4 \\ 0.2 & -1.4 \end{bmatrix} \begin{bmatrix} -\frac{9}{15} \\ \frac{13}{15} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 2 \\ \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$$

(Note that since there are cases, specifically when $\mathbf{b} \in Col(\mathbf{A})$, when exact solution is possible, these values **must** be computed!) However, in this case, this calculation shows that $\mathbf{x}$ is not an exact solution. But, since this is a least square problem, that is not unexpected. It simply means that $\mathbf{b} \notin Col(\mathbf{A})$. But we are assured, by the way that we've constructed this, that

$$\mathbf{r} = (\mathbf{b} - \mathbf{A}\mathbf{x}) \perp Col(\mathbf{A}) = Col(\mathbf{U})$$

which is all we can expect in a least squares solution.

This can be easily checked, since

$$\mathbf{r} = \mathbf{b} - \mathbf{A}\,\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ 2 \\ \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{3}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

and direct calculation shows

$$\mathbf{U}^H\,\mathbf{r} = \begin{bmatrix} \frac{5}{6} & \frac{3}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{3}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \mathbf{0}$$

3. (15 points) Show that if $\mathbf{Q}$ is any unitary matrix, and $\mathbf{x}$ and $\mathbf{y}$ are any two vectors, then the angle between $\mathbf{Qx}$ and $\mathbf{Qy}$ is the same as the angle between $\mathbf{x}$ and $\mathbf{y}$.

solution:

By definition, if $\mathbf{Q}$ is unitary, then

$$\mathbf{Q}^H \mathbf{Q} = \mathbf{I}$$

But also, in $\mathbb{C}^n$, by definition, the angles between $\mathbf{x}$ and $\mathbf{y}$, and between $\mathbf{Q}\,\mathbf{x}$ and $\mathbf{Q}\,\mathbf{y}$ satisfy, respectively

$$\cos(\theta) = \frac{\mathbf{x}^H\,\mathbf{y}}{\sqrt{\mathbf{x}^H\,\mathbf{x}}\sqrt{\mathbf{y}^H\,\mathbf{y}}} \quad \text{and} \quad \cos(\phi) = \frac{(\mathbf{Q}\,\mathbf{x})^H\,(\mathbf{Q}\,\mathbf{y})}{\sqrt{(\mathbf{Q}\,\mathbf{x})^H\,(\mathbf{Q}\,\mathbf{x})}\sqrt{(\mathbf{Q}\,\mathbf{y})^H\,(\mathbf{Q}\,\mathbf{y})}}$$

But also, in general, $(\mathbf{A}\,\mathbf{B})^H = \mathbf{B}^H\,\mathbf{A}^H$, and therefore

$$\cos(\phi) = \frac{\mathbf{x}^H\,\overbrace{\mathbf{Q}^H\mathbf{Q}}^{\mathbf{I}}\,\mathbf{y}}{\sqrt{\mathbf{x}^H\,\underbrace{\mathbf{Q}^H\mathbf{Q}}_{\mathbf{I}}\,\mathbf{x}}\sqrt{\mathbf{y}^H\,\underbrace{\mathbf{Q}^H\mathbf{Q}}_{\mathbf{I}}\,\mathbf{y}}} = \frac{\mathbf{x}^H\mathbf{y}}{\sqrt{\mathbf{x}^H\mathbf{x}}\sqrt{\mathbf{y}^H\mathbf{y}}} = \cos(\theta)$$

Therefore

$$\cos(\phi) = \cos(\theta) \quad \Longrightarrow \quad \theta = \phi$$

4. (20 points) a. A certain PC executes a critical MATLAB program far more slowly than would be expected based on the computational complexity of the problem and the CPU speed.

(1) If doubling the system RAM significantly speeds up the execution time, explain *briefly* the most probable explanation for the original slowness.

solution:

Almost certainly, the problem here was that, because of insufficient RAM, the system was having to page to and from "virtual" memory (swap), which was likely located on a hard (or even worse an network) disk. Since reading or writing data or program commands stored in swap requires both electronic and physical operations, such operations are very time-consuming. Adding RAM allowed the paging to be done to RAM, vice swap, and reading from or writing to RAM is usally several orders of magnitude faster than reading from or writing to disk.

(2) If doubling the system RAM does **not** significantly speed up the execution time, *briefly* identify the probable explanation(s) for the original slowness.

solution:

Since adding RAM did not speed up the code, then any paging was almost certainly already being done to RAM. Therefore, the slow execution of the code itself is almost certainly due to the either CPU spending the bulk of its time on non-computational operations, or its being idle while waiting for data to be paged in from RAM. The former would most likely come from inefficient code, e.g. code written using explicit double (or triple) loops, while the latter would most likely be due to some computationally intensive part of the code being written a in row-oriented, rather than column-oriented manner.

b. Assume that, for a given matrix $\mathbf{A}$, the MATLAB command

$$[\ \mathbf{Q}\ ,\ \mathbf{R}\ ] = \mathbf{qr(A)}$$

produces a $\mathbf{Q}$ and a $\mathbf{R}$ which satisfy, upon checking

$$\frac{\|\,\mathbf{A}-\mathbf{QR}\,\|}{\|\,\mathbf{A}\,\|} \doteq 1.3 \times 10^{-15}$$

Approximately how many digits of accuracy can you expect in the solution to $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, when computed by the MATLAB command:

$$\mathbf{x} \ = \ \mathbf{R} \backslash\ (\mathbf{Q}' {*} \mathbf{b})$$

solution:

The fundamental relationship between (backward) stability and condition is that, if an algorithm $(\tilde{f})$ is backward stable, i.e. if given a problem (f) with data $\mathbf{x}$, there always exists other data $(\tilde{\mathbf{x}})$ and a constant C such that

$$\frac{\|\,\mathbf{x}-\tilde{\mathbf{x}}\,\|}{\|\,\mathbf{x}\,\|} \le C\epsilon_{machine} \tag{1}$$

then the error in a computed solution for that data satisfies

$$\frac{\|\,f(\mathbf{x})-\tilde{f}(\mathbf{x})\,\|}{\|\,f(\mathbf{x})\,\|} \le C\kappa(\mathbf{x})\epsilon_{machine} \tag{2}$$

In this case, where the data is actually the matrix $\mathbf{A}$ and right-hand side vector $\mathbf{b}$, unfortunately

$$\frac{\|\,\mathbf{A}-\mathbf{QR}\,\|}{\|\,\mathbf{A}\,\|} \doteq 1.3 \times 10^{-15}$$

is simply an instance of equation (1), i.e. verifying only backward stability of the **QR algorithm** for the matrix $\mathbf{A}$. That result, however, tells us absolutely **nothing** about the condition of $\mathbf{A}$. Therefore, we have absolutely no idea how many digits, if any, of a computed solution to the **problem**

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

will be accurate.